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APPLICATION OF MELLIN TRANSFORM IN THE SOLUTION OF STEADY- STATE BOUNDARY VALUE PROBLEMS

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ABSTRACT

Besides the use of complex variable techniques in the solution of Dirichlet harmonic problems, the Mellin transform method is a variable tool in the solution and analysis of two dimensional harmonic Dirichlet problems in the theory of heat transfer, Dirichlet problems in ideal fluid flows, thermal physics, etc. In this paper, we obtained the steady state temperature distribution function in a shaded semi-infinite slab using the Mellin transform method. The problem is formulated using the conformal mapping. Due to the nature of the boundary condition, methods such as Fourier transform, D'Alembert solution and separation of variables cannot easily solve the problem. The steady state temperature distribution function in closed form obtained was seen to depend on the X , Y - coordinates.

Keywords: Mellin transform, steady-state temperature, Dirichlet problems, upper plane.

1. INTRODUCTION

Most real-life problems when formulated mathematically lead to partial differential equations together with the associated boundary conditions. The endeavour or problem of determining solutions to a partial differential equation which satisfy the boundary conditions is called a boundary –value problem. From the mathematical as well as physical point of view, it is of fundamental importance to ensure the existence and uniqueness of these solutions.

A survey of works in literature shows that boundary –value problems of complex geometry has been carried out by many researchers using different methods such as finite difference method [1,2], finite element technique [3,4], Wiener-Hopf techniques [5 ,6], integral transform method [7,8], variational methods [9],

Without loss of generality, from the complex variable technique, we know that complex functions provide an almost inexhaustible supply of harmonic functions, that is solutions to the two- dimensional Laplace's equations. Thus, to resolve such an associated

boundary –value problem, we seek the complex function whose real part matches the prescribed boundary conditions. Unfortunately, even for relatively simple domains, this remains a daunting task because of identifying the mapping function to use for a particular problem.

This aforementioned difficulty is highly avoided in the application of Mellin transform method [10] in that for a given boundary –value problem with complex boundary conditions, we employ the conformal mapping techniques which maps the complex domain in the Z plane to an upper half W -plane. This renders the problem amenable to method of integral transform.

2. THE FORMULATION OF THE PROBLEM

Consider a semi-infinite slab (shaded in fig.1) of width a units, with its boundaries maintained at the indicated temperatures where T is constant. The problem is to find steady-state temperature in the shaded region

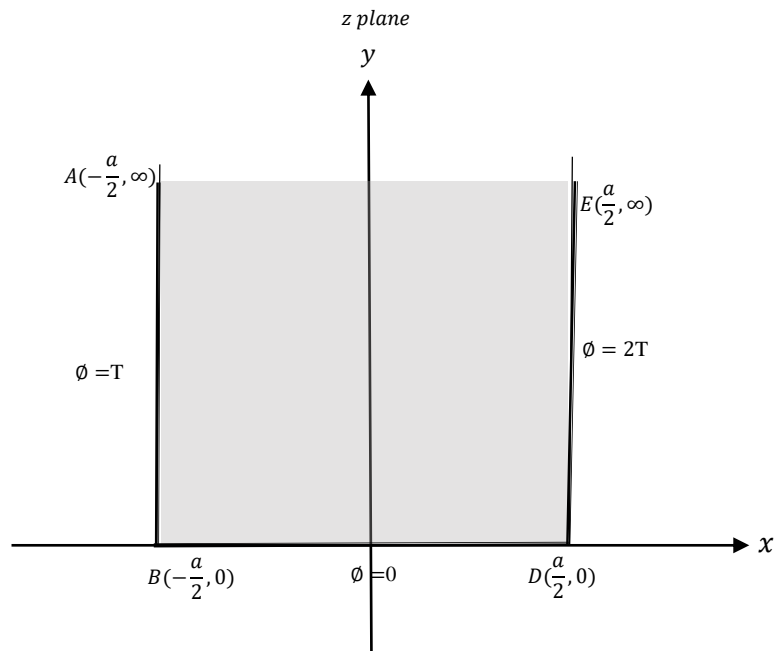


Fig.1: A semi-infinite shaded slab in the z-plane with the loading sites
Mathematically in the rectangular coordinate we are solving a Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad -\frac{1}{2} < x < \frac{1}{2}, 0 \leq y < \infty \quad (1)$$

Subjected to the boundary conditions

$$\phi\left(-\frac{a}{2}, y\right) = T \quad 0 \leq y < \infty \quad (2)$$

$$\phi\left(\frac{a}{2}, y\right) = 2T \quad 0 \leq y < \infty \quad (3)$$

$$\phi(x, 0) = 0 \quad -\frac{a}{2} \leq x < \frac{a}{2} \quad (4)$$

3.0 TRANSFORMATION OF THE PROBLEM

Before now the separation of variable method has been used to solve problems with simple geometry. But complex geometries are hardly handled by it, hence our method achieves this feat by using conformal

mapping which makes the problem analyzable by method of integral transformation. Thus, the original Z plane of analysis is transformed onto the upper half of the W plane using the mapping function

$$w(z) = \sin\left(\frac{\pi z}{a}\right), \quad z = x + iy \quad (5)$$

Let $w = u + iv$ i.e. $u + iv = \sin\frac{\pi}{a}(x + iy)$ (6)

Which is equivalent to

$$u = \sin\left(\frac{\pi x}{a}\right) \cosh\left(\frac{\pi y}{a}\right), \quad v = \cos\left(\frac{\pi x}{a}\right) \sinh\left(\frac{\pi y}{a}\right) \quad (7)$$

Using eqn. (7) and the corresponding points we the transform plane as

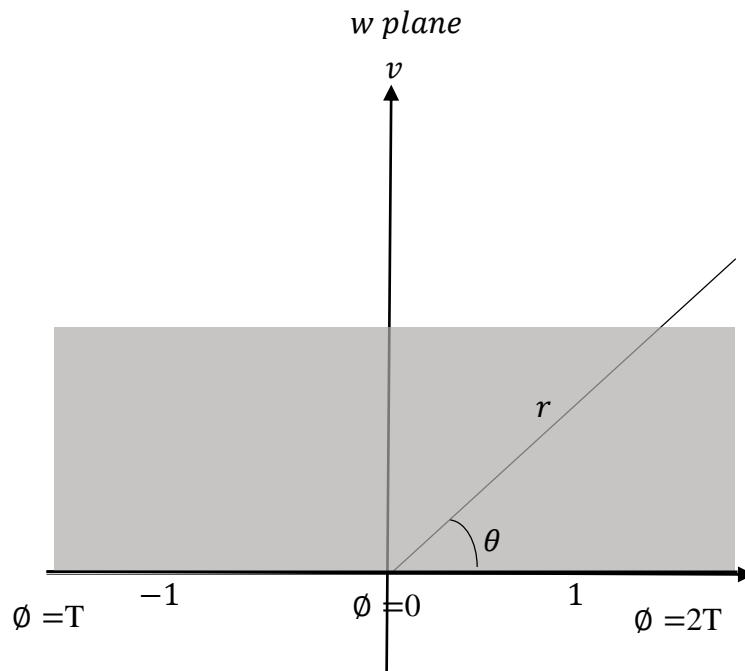


Fig. 2: showing boundary conditions in the upper half plane of the w-plane.

From the conformality condition, we have

$$\phi(x, y) = w(u, v) = w(r, \theta) = w\left(e^{\frac{\pi x}{a}}, \frac{\pi}{a} y\right) \quad (8)$$

Hence in polar form, the boundary value problem becomes

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = 0 \quad (9)$$

$$w\left(r, -\frac{\pi}{2}\right) = T \quad (10)$$

$$w\left(r, \frac{\pi}{2}\right) = 2T \quad (11)$$

$$w(r, 0) = 0 \quad (12)$$

We now introduce the Mellin integral transform defined by

$$\bar{w}(s, \theta) = \int_0^{\infty} w(r, \theta) r^{s-1} dr \quad (13)$$

While the inverse Mellin transform of $\bar{w}(s, \theta)$ is define by [11]

$$w(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{w}(s, \theta) r^{-s} ds \quad (14)$$

Next, we rewriting eqn.(9) before applying the Mellin transform, we have

$$r^2 \frac{\partial^2 w(r, \theta)}{\partial r^2} + r \frac{\partial w(r, \theta)}{\partial r} + \frac{\partial^2 w(r, \theta)}{\partial \theta^2} = 0 \quad (15)$$

This implies

$$r \frac{\partial}{\partial r} \left(r \frac{\partial w(r, \theta)}{\partial r} \right) + \frac{\partial^2 w(r, \theta)}{\partial \theta^2} = 0 \quad (16)$$

Applying the Mellin transform to eqn. (16), we have

$$\int_0^{\infty} r \frac{\partial}{\partial r} \left(r \frac{\partial w(r, \theta)}{\partial r} \right) r^{s-1} \partial r + \int_0^{\infty} \frac{\partial^2 w(r, \theta)}{\partial \theta^2} r^{s-1} \partial r = 0 \quad \text{Mellin} \quad (17)$$

Evaluating the first integral using integration by parts and the second integral by the Leibnitz rule, we have

$$\frac{d^2 \bar{w}(s, \theta)}{d\theta^2} + s^2 \bar{w}(s, \theta) = 0 \quad (18)$$

We also apply the Mellin transform to the boundary conditions as follows

$$\bar{w}\left(s, -\frac{\pi}{2}\right) = \int_0^1 r^{s-1} (T) dr + \int_1^{\infty} r^{s-1} (0) dr = \frac{T}{s} + 0 = \frac{T}{s} \quad (19)$$

Similarly,

$$\bar{w}\left(s, \frac{\pi}{2}\right) = \int_0^1 r^{s-1} (2T) dr + \int_1^{\infty} r^{s-1} (0) dr = \frac{2T}{s} + 0 = \frac{2T}{s} \quad (20)$$

4.0 Solution of the transformed problem.

In solving eqn (18), we assume a solution of the form

$$\bar{w}(s, \theta) = A \sin \theta + B \cos s\theta \quad (21)$$

Where the value of the coefficients A and B are obtained using the boundary conditions (19), (20) as

$$\bar{w}\left(s, -\frac{\pi}{2}\right) = -A \sin \frac{\pi}{2} s + B \cos \frac{\pi}{2} s = \frac{T}{s} \quad (22)$$

$$\bar{w}\left(s, \frac{\pi}{2}\right) = A \sin \frac{\pi}{2} s + B \cos \frac{\pi}{2} s = \frac{2T}{s} \quad (23)$$

Solving equations (22) and (23), we have

$$A = \frac{T}{2s \sin \frac{\pi}{2} s}, \quad B = \frac{3T}{2s \cos \frac{\pi}{2} s} \quad (24)$$

Substituting eqn. (24) into eqn. (21), we have

$$\bar{w}(s, \theta) = \frac{T}{2s \sin \frac{\pi}{2} s} \sin \theta + \frac{3T}{2s \cos \frac{\pi}{2} s} \cos s\theta \quad (25)$$

Next, we apply the inverse Mellin transform eqn. (14) to eqn. (25) to obtain the potential steady- state temperature distribution sought for as

$$w(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s} \left(\frac{T}{2s \sin \frac{\pi}{2} s} \sin \theta + \frac{3T}{2s \cos \frac{\pi}{2} s} \cos s\theta \right) ds \quad (26)$$

The contour integral in eqn. (26) can be evaluated using Cauchy's residue theorem [12]

= residue of $\bar{w}(s, \theta)$ evaluated at the poles

The poles of $\bar{w}(s, \theta)$ are simple at $s = \pm 2n$, $n = 1, 2, 3, \dots$ while the double poles are located at $s = 0$

At the pole $s = 0$, we have

$$\text{Residue} = \lim_{s \rightarrow 0} \bar{w}(s, \theta) r^{-s} = \lim_{s \rightarrow 0} \left(\frac{T}{2} \left[\frac{s}{s \sin \frac{\pi}{2} s} \right] \sin s\theta + \frac{3Ts}{2s} \frac{\cos s\theta}{\cos \frac{\pi}{2} s} \right) r^{-s}$$

$$\begin{aligned}
&= \lim_{s \rightarrow 0} \left(\frac{T}{2} \left[\frac{s}{\sin \frac{\pi}{2}s} \frac{\sin s\theta}{s} + \frac{3T}{2} \frac{\cos s\theta}{\cos \frac{\pi}{2}s} \right] r^{-s} \right) \\
&= \lim_{s \rightarrow 0} \left(\frac{T}{2} \left[\frac{\frac{\pi}{2}s}{\frac{\pi}{2} \sin \frac{\pi}{2}s} \frac{\theta \sin s\theta}{\theta s} + \frac{3T}{2} \frac{\cos s\theta}{\cos \frac{\pi}{2}s} \right] r^{-s} \right) \\
&= \lim_{s \rightarrow 0} \frac{T}{2} \left(\frac{2}{\pi} (1)\theta(1) + \frac{3T}{2} \frac{\cos s0}{\cos \frac{\pi}{2}0} \right) r^{-0} \\
&= \frac{3T}{2} + \frac{T}{\pi} \theta \tag{27}
\end{aligned}$$

At the pole $s = -2n$, $n = 1, 2, 3, \dots$, we have

$$\begin{aligned}
\text{Residue} &= \lim_{s \rightarrow -2n} (s + 2n) \bar{w}(s, \theta) r^{-s} = \lim_{s \rightarrow -2n} \left(\frac{T}{2} \left[\frac{s + 2n}{\sin \frac{\pi}{2}s} \frac{\sin s\theta}{s} + \frac{3T}{2} \frac{(s + 2n) \cos s\theta}{s \cos \frac{\pi}{2}s} \right] r^{-s} \right) \\
&= \lim_{s \rightarrow -2n} \left(\frac{2T}{2\pi} \left[\frac{1}{\cos \frac{\pi}{2}(-2n)} \frac{\sin(-2n)\theta}{(-2n)} + 0 \right] r^{2n} \right) \\
&= \frac{T}{2\pi} \left[\left(\frac{1}{\cos n\pi} \right) \left(\frac{\sin 2n\theta}{n} \right) \right] r^{2n} \\
&= \frac{T}{2\pi} \left[(-1)^n \frac{\sin 2n\theta}{n} \right] r^{2n} \tag{28}
\end{aligned}$$

At the pole $s = 2n$, $n = 1, 2, 3, \dots$, we have

$$\begin{aligned}
\text{Residue} &= \lim_{s \rightarrow 2n} (s - 2n) \bar{w}(s, \theta) r^{-s} = \lim_{s \rightarrow 2n} \left(\frac{T}{2} \left[\frac{s - 2n}{\sin \frac{\pi}{2}s} \frac{\sin s\theta}{s} + \frac{3T}{2} \frac{(s - 2n) \cos s\theta}{s \cos \frac{\pi}{2}s} \right] r^{-s} \right) \\
&= \lim_{s \rightarrow 2n} \left(\frac{2T}{2\pi} \left[\frac{1}{\cos \frac{\pi}{2}(2n)} \frac{\sin(2n)\theta}{(2n)} + 0 \right] r^{-2n} \right) \\
&= \frac{T}{2\pi} \left[\left(\frac{1}{\cos n\pi} \right) \left(\frac{\sin 2n\theta}{n} \right) \right] r^{2n}
\end{aligned}$$

$$= \frac{T}{2\pi} \left[(-1)^n \frac{\sin 2n\theta}{n} \right] r^{-2n} \quad (29)$$

Hence

$$w(r, \theta) = \begin{cases} \left[\frac{3T}{2} + \frac{T}{\pi} \theta + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] \right] r^{2n} & , r < 1 \\ \left[\frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] \right] r^{-2n} & , r > 1 \end{cases} \quad (30)$$

This is the steady-state temperature distribution (displacement) function sought for in polar form.

5.0 Results and discussion

Under this section, we undertake a verification of our result in eqn. (30) to ascertain if it satisfies eqns. (9),(10) and (11), i.e., the governing equation and its boundary conditions.

For $r < 1$

$$\begin{aligned} w(r, \theta) &= \frac{3T}{2} + \frac{T}{\pi} \theta + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n} \\ \frac{\partial w(r, \theta)}{\partial r} &= (2n) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n-1} \\ \frac{\partial^2 w(r, \theta)}{\partial r^2} &= 2n(2n-1) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n-2} \\ \frac{\partial w(r, \theta)}{\partial \theta} &= \frac{T}{\pi} + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(2n)(-1)^n \cos 2n\theta}{n} \right] r^{2n} \\ \frac{\partial^2 w(r, \theta)}{\partial \theta^2} &= -(2n)^2 \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n} \\ \therefore \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} &= 2n(2n-1) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n-2} + \\ (2n) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n-2} - (2n)^2 \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n-2} &= 0 \end{aligned} \quad (31)$$

For $r > 1$

$$\begin{aligned} w(r, \theta) &= \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n} \\ \frac{\partial w(r, \theta)}{\partial r} &= -(2n) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n-1} \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 w(r, \theta)}{\partial r^2} &= -(2n)(-2n-1) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n-2} \\
\frac{\partial w(r, \theta)}{\partial \theta} &= (2n) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos 2n\theta}{n} \right] r^{-2n} \\
\frac{\partial^2 w(r, \theta)}{\partial \theta^2} &= -(2n)^2 \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n} \\
\therefore \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} &= -(2n)(-2n-1) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n-2} \\
&\quad - (2n) \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n-2} - (2n)^2 \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n-2} = 0 \quad (32)
\end{aligned}$$

The eqn. (30) has satisfied the Laplace equation in (9) for both $r < 1$ and $r < 1$ respectively. Verification of the boundary conditions (10), (11).

For $r < 1$

$$\begin{aligned}
w(r, \theta) &= \frac{3T}{2} + \frac{T}{\pi} \theta + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n} \\
w\left(r, -\frac{\pi}{2}\right) &= \frac{3T}{2} + \frac{T}{\pi} \left(-\frac{\pi}{2}\right) + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\left(-\frac{\pi}{2}\right)}{n} \right] r^{2n} \\
&= \frac{3T}{2} - \frac{T}{2} - \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin n\pi}{n} \right] r^{2n} \\
&= \frac{3T}{2} - \frac{T}{2} - \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n (0)}{n} \right] r^{2n} \\
&= \frac{3T}{2} - \frac{T}{2} - 0 = T \quad (33)
\end{aligned}$$

For $r > 1$

$$w(r, \theta) = \frac{3T}{2} + \frac{T}{\pi} \theta + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{2n}$$

$$\begin{aligned}
w\left(r, \frac{\pi}{2}\right) &= \frac{3T}{2} + \frac{T}{\pi}\left(\frac{\pi}{2}\right) + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\left(\frac{\pi}{2}\right)}{n} \right] r^{2n} \\
&= \frac{3T}{2} + \frac{T}{2} + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin n\pi}{n} \right] r^{2n} \\
&= \frac{3T+T}{2} + \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n (0)}{n} \right] r^{2n} \\
&= \frac{4T}{2} + 0 = 2T
\end{aligned} \tag{34}$$

For $r > 1$

$$\begin{aligned}
w(r, \theta) &= \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\theta}{n} \right] r^{-2n} \\
w\left(r, \frac{\pi}{2}\right) &= \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin 2n\left(\frac{\pi}{2}\right)}{n} \right] r^{-2n} \\
&= \frac{T}{2\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \sin n\pi}{n} \right] r^{-2n} = 0
\end{aligned} \tag{35}$$

Eqn. (30) has also satisfied the boundary conditions (10),(11) for $r < 1$ and $r > 1$ in polar coordinates respectively.

6.0 Conclusion

Following the verification systematically, it is not a difficult algebra to show that our eqn.(30) satisfies both the Laplace's equation and the boundary

equation for $e^{\frac{\pi}{a}x} < 1$ and $e^{\frac{\pi}{a}x} > 1$ which invariably depends on x and y . This shows eqn. (30) is indeed the potential steady-state temperature distribution function we are looking for because it has satisfied both the Laplace's equation and the boundary equations. Recall that the problem was to find the steady-state temperature distribution function in an infinite slab of thickness of a units. In order to solve this problem, we use the known idea [11] that the temperature distribution function satisfies the Laplace's equation in the interior of the slab.

The boundary conditions are such that the Dirichlet problem that models the temperature distribution function cannot be solved by methods of separation of variables method and D'Alembert solution, hence the region representing the slab has to conformally transformed unto the upper half plane to make the boundaries suitable for application of an integral transform method, which in this study, is the Mellin transform method.

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